J. Fluid Mech. (1965), *vol.* 23, *part* 4, *pp.* 625–643
Printed in Great Britain

The steady motion of a particle of arbitrary shape at small Reynolds numbers

By R. G. COX
Department of Chemical Engineering, New York University,
New York 53, N.Y., U.S.A.

(Received 16 April 1965)

The results given by Brenner & Cox (1963) for the resistance of a particle of arbitrary shape in translation at small Reynolds numbers are generalized. Thus we consider here a single particle of arbitrary shape moving with both translation *and rotation* in an infinite fluid, the Reynolds number R of the fluid motion being assumed small. With the additional assumption that the motion is steady with respect to some inertial frame of reference, we calculate both the force *and couple* on the body as an expansion in the Reynolds number to $O(R^2 \ln R)$. This force and couple are expressed entirely in terms of various Stokes flows for the given body in rotation or translation.

A discussion is given of the form taken by the formulae for the force and couple for cases in which the body possesses symmetry properties. Quantitative results are obtained for both a spheroid and a dumb-bell-shaped body in pure translation and also for a translating rotating sphere and for a dumb-bell-shaped body in pure rotation.

The application of the general results to 'quasi-steady' problems is considered, with particular reference to a freely falling spheroid (of small eccentricity) which is shown to orientate itself so that it is broad-side on to its direction of motion.

Finally the general results are compared with those that would be obtained by the use of the Oseen equations. By consideration of a particular example it is shown that the Oseen equations do not in general give the correct value of the force on the body to $O(R)$.

1. Introduction

We consider a solid body of arbitrary shape undergoing uniform translation and/or rotation in an infinite incompressible fluid, the Reynolds number (R) of the fluid motion being small. For particular body shapes, the value of the force and couple acting on the body have been given as an expansion in the Reynolds number to $O(R^2 \ln R)$. Thus Proudman & Pearson (1957) and Breach (1961) have respectively calculated the force acting on a sphere and an ellipsoid in pure translation, whilst Rubinow & Keller (1961) have given the force and couple acting upon a rotating translating sphere, the direction of translation being perpendicular to the axis of rotation. Brenner & Cox (1963) gave the force to $O(R^2 \ln R)$ on a body of *arbitrary* shape in *pure translation*, giving their result in terms of the Stokes flows for the body in translation in any three non-coplanar directions. We consider here the more general problem of the calculation of the force **F** *and couple* **G** on a body of arbitrary shape which is undergoing a general motion consisting of a translation and/or a rotation, subject only to the restriction that the resulting fluid motion be steady relative to some inertial frame of reference. The computation of this force and couple to $O(R^2 \ln R)$ requires only the knowledge of: (i) the Stokes flow fields due to the given body being translated in any three non-coplanar directions in a fluid at rest at infinity; and (ii) the Stokes flow fields due to the given body being rotated about any three non-coplanar directions in a fluid at rest at infinity.

As in the case of bodies undergoing purely translational motion (Brenner & Cox 1963), it is shown that these formulae take on a more simple form for bodies possessing symmetry properties, the force and couple then being expressible in terms of the forces and/or couples on the body for the Stokes flows (i) and (ii), and not in terms of the details of these velocity fields.

Two general classes of problems are considered in more detail. The first of these is that of finding the force *and couple* on a non-rotating body of arbitrary shape, for which it is shown that the value of the force is the same as that obtained by Brenner & Cox (1963). The second class of problems is that of the rotating axially symmetric body in uniform translation.

As examples of the problem concerning bodies in pure translation, we consider that of the uniformly translating spheroid and that of the uniformly translating dumb-bell for which explicit formulae are obtained for the force and couple. In the former example, it is shown that there exists, in general, a couple of $O(R)$ acting on centrally symmetric bodies, whilst in the latter it is shown that the force on a body to $O(R)$ is not in general reversed by a reversal of the body translation velocity.

For the class of rotating axially symmetric bodies in translation, we consider as specific examples that of the rotating translating sphere and that of an axisymmetric dumb-bell-shaped body in *pure* rotation about its axis of symmetry. In the former example, we obtain a more general form of the results obtained by Rubinow & Keller (1961) concerning a rotating translating sphere, and in the latter we demonstrate that for an axially symmetric body without fore-aft symmetry, a *pure* rotation about the symmetry axis can, in general, produce a force of $O(R)$ on the body directed along its axis, although the body is not in translation. Such a force is invariant to the direction of rotation.

A short discussion is then given of the applicability of the general theory to non-steady problems for which it is shown that, in considering the fall under gravity of a body of arbitrary shape through a fluid, the positions of equilibrium orientation may be found to $O(R)$ from the theory although the general motion may not be steady. For such a problem, the general formulae derived for steady motion cannot be used to predict the *complete* body motion, except for very special cases for which the body motion may be completely 'quasi-steady'. One example of such 'quasi-steady' motion is that of the free fall of a spheroidal body of small eccentricity. This problem is considered in detail, it being shown that such a body takes up a stable equilibrium orientation with its axis horizontal or vertical according to whether it is prolate or oblate.

Finally, formulae are given for the force and couple to $O(R)$ on a non-rotating body in uniform translation as calculated by the Oseen equations. It is shown by the consideration of a dumb-bell-shaped body that the force on a body as calculated by the use of these equations is, in general, incorrect to $O(R)$. This rectifies an error made in Appendix 2 of the paper by Brenner & Cox (1963), where an attempt was made to prove the above statement by the consideration of a slightly deformed sphere.

2. The inner and outer expansions

We consider a body with a surface B' of arbitrary shape rotating and/or translating in an incompressible fluid at rest at infinity in such a manner that the motion is steady relative to some inertial frame of reference. Thus we restrict ourselves to the following two classes of problems: *class* (*a*) bodies of arbitrary shape in pure uniform translation; and *class* (*b*) axially symmetric bodies in uniform rotation about their axes, together with a uniform translation in some arbitrary direction.

By choosing an origin of co-ordinates O fixed in the body and lying on the axis of symmetry for *class* (*b*) [or anywhere within the body for *class* (*a*)], we may define the (dimensional) velocity $\mathbf{V}'$ and angular velocity $\mathbf{\Omega}'$ as being respectively the velocity of O and the angular velocity of the body relative to non-rotating axes moving with O.

Define c to be a characteristic body dimension and U to be any characteristic velocity which we could take to be equal to either $|\mathbf{V}'|$ or $|c\mathbf{\Omega}'|$. The fluid velocity $\mathbf{u}'$ and pressure p', together with the general position vector $\mathbf{r}'$ (taken relative to O) may then be expressed as dimensionless (undashed) quantities by the relations

$$\mathbf{u} = \mathbf{u}'/U, \quad p = (c/\mu U)(p' - p'_\infty), \quad \mathbf{r} = \mathbf{r}'/c, \tag{2.1}$$

where p'_∞ is the constant pressure at infinity.

The equations of general fluid motion relative to O may then be put in the dimensionless form

$$\nabla^2\mathbf{u} - \nabla p = R\mathbf{u}.\nabla\mathbf{u}, \quad \nabla.\mathbf{u} = 0, \tag{2.2}$$

with the boundary conditions

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{\Omega}\wedge\mathbf{r} \quad \text{on} \quad B, \\ \mathbf{u} &\to -\mathbf{V} \quad \text{as} \quad r\to\infty, \end{aligned}\right\} \tag{2.3}$$

where $\mathbf{V}$ and $\mathbf{\Omega}$ are the dimensionless body velocity and angular velocity given by

$$\mathbf{V} = \mathbf{V}'/U, \quad \mathbf{\Omega} = \mathbf{\Omega}'(c/U). \tag{2.4}$$

B is the body surface expressed in dimensionless form and R the Reynolds number defined as $(cU\rho/\mu)$.

In solving equations (2.2) with boundary conditions (2.3) we proceed by forming inner and outer expansions in a manner similar to that done by Brenner & Cox (1963) for a body in pure translation. The details of this procedure are therefore not given here.

The velocity and pressure fields may be expanded in the neighbourhood of the body as

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{u}_0 + R\mathbf{u}_1 + o(R), \\ p &= p_0 + Rp_1 + o(R), \end{aligned}\right\} \tag{2.5}$$

which we call the *inner* expansion. $(\mathbf{u}_0, p_0)$ satisfies

$$\left.\begin{aligned} \nabla^2\mathbf{u}_0 - \nabla p_0 = \mathbf{0}, \quad \nabla . \mathbf{u}_0 = 0, \\ \mathbf{u}_0 = \mathbf{\Omega} \wedge \mathbf{r} \quad \text{on} \quad B, \end{aligned}\right\} \tag{2.6}$$

whilst $(\mathbf{u}_1, p_1)$ satisfies

$$\left.\begin{aligned} \nabla^2\mathbf{u}_1 - \nabla p_1 = \mathbf{u}_0 . \nabla \mathbf{u}_0, \quad \nabla . \mathbf{u}_1 = 0, \\ \mathbf{u}_1 = \mathbf{0} \quad \text{on} \quad B. \end{aligned}\right\} \tag{2.7}$$

Defining *outer* variables by the relation

$$\tilde{\mathbf{r}} = R\mathbf{r}, \tag{2.8}$$

the outer expansion may be seen to be of the form

$$\left.\begin{aligned} \mathbf{u} &= -\mathbf{V} + R\mathbf{U}_1 + o(R), \\ p &= R^2 P_1 + o(R^2), \end{aligned}\right\} \tag{2.9}$$

where $(\mathbf{U}_1, P_1)$ satisfies

$$\left.\begin{aligned} \tilde{\nabla}^2\mathbf{U}_1 + \tilde{\nabla} P_1 = -\mathbf{V} . \tilde{\nabla}\mathbf{U}_1, \quad \tilde{\nabla} . \mathbf{U}_1 = 0 \\ \mathbf{U}_1 \to \mathbf{0} \quad \text{as} \quad \tilde{r} \to \infty. \end{aligned}\right\} \tag{2.10}$$

The inner boundary conditions on $(\mathbf{U}_1, P_1)$ and the outer boundary conditions on $(\mathbf{u}_0, p_0)$ and $(\mathbf{u}_1, p_1)$ are determined by the required matching of the two expansions.

It may be shown that the asymptotic expansion for large r of the zeroth-order inner approximation $(\mathbf{u}_0, p_0)$ is

$$\left.\begin{aligned} \mathbf{u}_0 &= -\mathbf{V} - \tfrac{3}{4}[\mathbf{s}(\mathbf{F}_0)] + \lambda_{pq}\mathbf{i}_p . \nabla[\mathbf{s}(\mathbf{i}_q)], \\ p_0 &= -\tfrac{3}{4}[t(\mathbf{F}_0)] + \lambda_{pq}\mathbf{i}_p . \nabla[t(i_q)], \end{aligned}\right\} \tag{2.11}$$

where $[\mathbf{s}(\mathbf{a})]$ and $[t(\mathbf{a})]$ are defined by

$$\left.\begin{aligned} [\mathbf{s}(\mathbf{a})] &= (\mathbf{a} + \mathbf{r} . \mathbf{a}\mathbf{r}/r^2)/r, \\ [t(\mathbf{a})] &= 2\mathbf{a} . \mathbf{r}/r^3, \end{aligned}\right\} \tag{2.12}$$

$\mathbf{i}_p$, $\mathbf{i}_q$ being unit vectors in the p and q Cartesian directions and $\mathbf{F}_0$ the Stokes force on the body. λ_{pq} is a second-order tensor dependent only upon the shape of the body.

The first-order outer approximation $(\mathbf{U}_1, P_1)$ may then be determined from the equations (2.10) together with the required matching conditions as

$$\left.\begin{aligned} \mathbf{U}_1 = -\frac{3}{4\tilde{r}}\left\{\mathbf{F}_0 + \frac{1}{\tilde{r}^2}(\tilde{\mathbf{r}} . \mathbf{F}_0\tilde{\mathbf{r}})\right\}\exp\left[-\tfrac{1}{2}(V\tilde{r} + \mathbf{V} . \tilde{\mathbf{r}})\right] + \frac{3}{2V}\{1 - [1 + \tfrac{1}{2}(V\tilde{r} + \tilde{\mathbf{r}} . \mathbf{V})] \\ \times \exp\left[-\tfrac{1}{2}(V\tilde{r} + \mathbf{V} . \tilde{\mathbf{r}})\right]\}(\mathbf{F}_0 . \tilde{\nabla})\tilde{\nabla}\ln(V\tilde{r} + \mathbf{V} . \tilde{\mathbf{r}}), \\ P_1 = -\tfrac{3}{2}\mathbf{F}_0 . \tilde{\mathbf{r}}/\tilde{r}^3. \end{aligned}\right\} \tag{2.13}$$

From the matching conditions, the outer boundary conditions on $(\mathbf{u}_1, p_1)$ may be obtained, from which the value of the asymptotic expansion for large r, of the solution of equations of (2.7) may be obtained as

$$\left.\begin{aligned} \mathbf{u}_1 = & -\tfrac{3}{16}(\mathbf{V}.\nabla)\,[r(-3\mathbf{F}_0+\mathbf{r}.\mathbf{F}_0\mathbf{r}/r^2)]+\tfrac{3}{16}(3\mathbf{F}_0V^2-\mathbf{V}.\mathbf{F}_0\mathbf{V})/V \\ & +\tfrac{1}{4}\lambda_{pq}(\mathbf{V}.\nabla)\,(\mathbf{i}_p.\nabla)\,[r(-3\mathbf{i}_q+\mathbf{r}.\mathbf{i}_q\mathbf{r}/r^2)] \\ & +\tfrac{9}{32}r^{-2}\{\mathbf{F}_0.\mathbf{r}\mathbf{F}_0-F_0^2\mathbf{r}+2(\mathbf{F}_0.\mathbf{r})^2\,\mathbf{r}/r^2\}+[\mathbf{s}(\boldsymbol{\delta})]+O(r^{-2}), \\ p_1 = & +\tfrac{9}{16}r^{-2}\{-F_0^2+2(\mathbf{F}_0.\mathbf{r})^2/r^2\}+[t(\boldsymbol{\delta})]+O(r^{-3}), \end{aligned}\right\} \quad (2.14)$$

where $\boldsymbol{\delta}$ is a constant vector determined by the boundary condition that $\mathbf{u}_1 = 0$ on the surface B of the body.

If we define the dimensionless force $\mathbf{F}$ and couple $\mathbf{G}$ acting on the body in terms of the corresponding dimensional quantities $\mathbf{F}'$ and $\mathbf{G}'$ by the relations

$$\mathbf{F} = \mathbf{F}'/6\pi\mu cU, \quad \mathbf{G} = \mathbf{G}'/6\pi\mu c^2U, \quad (2.15)$$

then it is seen that they may be written as

$$\mathbf{F} = \mathbf{F}_0+R\mathbf{F}_1+o(R), \quad \mathbf{G} = \mathbf{G}_0+R\mathbf{G}_1+o(R), \quad (2.16)$$

where $\mathbf{F}_0$ and $\mathbf{G}_0$ are the Stokes force and couple due to $(\mathbf{u}_0, p_0)$, with $\mathbf{F}_1$ and $\mathbf{G}_1$ likewise being due to $(\mathbf{u}_1, p_1)$.

Since $(\mathbf{u}_0, p_0)$ satisfies equations (2.6) with the outer boundary condition that $\mathbf{u}_0 \to -\mathbf{V}$ as $r \to \infty$, it is seen that $\mathbf{F}_0$ and $\mathbf{G}_0$ are related to $\mathbf{V}$ and $\boldsymbol{\Omega}$ (see Brenner 1963 and 1964*b*) in tensor notation, by

$$(F_0)_i = A_{ij}V_j+D_{ij}\Omega_j, \quad (G_0)_i = B_{ij}V_j+E_{ij}\Omega_j, \quad (2.17)$$

where
$$A_{ij} = A_{ji}, \quad E_{ij} = E_{ji}, \quad B_{ij} = D_{ji}. \quad (2.18)$$

3. First-order force and couple

The values of the first-order force $\mathbf{F}_1$ and couple $\mathbf{G}_1$ acting on the body need the knowledge of the asymptotic expansion for $(\mathbf{u}_1, p_1)$ to order (r^{-2}, r^{-3}) for large r. They cannot therefore be obtained directly from the expansions (2.14). We therefore proceed in a manner similar to that used by Brenner & Cox (1963).

Consider first the more general problem concerning *Stokes* flow $(\mathbf{v}, q)$ resulting from a given body force field $\mathbf{f}$ acting upon the fluid in which there is a solid body of arbitrary shape with surface B moving with a velocity $\mathbf{V}$ and angular velocity $\boldsymbol{\Omega}$ (relative to an origin O fixed in the body and moving with velocity $\mathbf{V}$). Thus

$$\nabla^2\mathbf{v}-\nabla q+\mathbf{f} = \mathbf{0}, \quad \nabla.\mathbf{v} = 0, \quad \mathbf{v} = \mathbf{V}+\boldsymbol{\Omega}\wedge\mathbf{r} \text{ on } B. \quad (3.1)$$

The following restrictions are placed on the vector function f_1: (i) $f_1 = O(r^{-2})$ as $r \to \infty$, and (ii) if f_i possesses a term, ${}_{-2}f_i$ say, in r^{-2} in its asymptotic expansion for large r, then this term changes sign upon replacing $\mathbf{r}$ by $-\mathbf{r}$. $\mathbf{v}$ is then of the form

$$\mathbf{v} = \mathbf{v}^*+\mathbf{k}+O(r^{-1}) \quad \text{as} \quad r \to \infty,$$

where $\mathbf{k}$ is a constant vector and $\mathbf{v}^*$ is a vector field which changes sign upon replacing $\mathbf{r}$ by $-\mathbf{r}$.

We now define $\bar{u}_{ij}$ to be a function of position defined as the ith component of the Stokes velocity field resulting from the pure translation of the body with unit velocity in the jth direction in a fluid at rest at infinity. Consider the flow field $\bar{u}_i = \bar{u}_{ij}\bar{V}_j$, where $\bar{V}_j$ is any arbitrary vector. This satisfies

$$\left.\begin{array}{l}\nabla^2\bar{\mathbf{u}} - \nabla\bar{p} = \mathbf{0}, \quad \nabla.\bar{u} = 0, \\ \bar{\mathbf{u}} = \bar{\mathbf{V}} \quad \text{on} \quad B, \quad \bar{\mathbf{u}}\to\mathbf{0} \quad \text{as} \quad r\to\infty.\end{array}\right\} \tag{3.2}$$

The quantity $\bar{u}_{ij}$ is a tensor function of position which may in fact be determined from the knowledge of the Stokes velocity fields resulting from the pure translation of the body *in any three* non-coplanar directions.

By taking the scalar product of (3.2) with $\mathbf{v}$ and that of (3.1) with $\bar{\mathbf{u}}$, subtracting and taking the volume integral of the resulting equation over the fluid contained within a large sphere, it may be shown that the force $\mathbf{F}$ on the body B due to the flow (3.1) is given by

$$F_i = A_{ij}(V_j - k_j) + D_{ij}\Omega_j + \frac{1}{6\pi}\int_\Gamma \bar{u}_{ji}f_j\,dV, \tag{3.3}$$

where Γ is the entire fluid volume. The details of this proof are given in a dissertation by the author (Cox 1964).

In a similar manner we define $\hat{u}_{ij}$ as the ith component of the Stokes velocity field resulting from the pure rotation of the body B with unit angular velocity about an axis in the jth direction in a fluid at rest at infinity. The couple $\mathbf{G}$ on the body due to the flow $(\mathbf{v}, q)$ may then be written

$$G_i = B_{ij}(V_j - k_j) + E_{ij}\Omega_j + \frac{1}{6\pi}\int_\Gamma \hat{u}_{ji}f_j\,dV. \tag{3.4}$$

The quantity $\hat{u}_{ij}$ is a tensor function of position and may be determined from a knowledge of the Stokes velocity fields resulting from the rotation of the body *about any three* non-coplanar axes.

Returning to the equation (2.7) it is seen that we may now find the force $\mathbf{F}_1$ and couple $\mathbf{G}_1$ in terms of $\bar{u}_{ij}$ and $\hat{u}_{ij}$ by the use of equations (3.3) and (3.4) with $\mathbf{f} = -(\mathbf{u}_0.\nabla\mathbf{u}_0)$, when it is noted that by the linearity of the Stokes equations, $\mathbf{u}_0$ itself may be written as

$$(u_0)_j = \hat{u}_{jp}\Omega_p + \bar{u}_{jp}V_p - V_j. \tag{3.5}$$

From the asymptotic form of $\mathbf{u}_1$ given by (2.14), it is seen that $\mathbf{k}$ must be taken to be

$$\mathbf{k} = \tfrac{3}{16}(3\mathbf{F}_0V^2 - \mathbf{V}.\mathbf{F}_0\mathbf{V})/V. \tag{3.6}$$

Thus, after some manipulation of the volume integrals and by the use of the asymptotic forms of $\mathbf{u}_0$, $\bar{u}_{ij}$, $\hat{u}_{ij}$ for large values of r (see Cox 1964), one finally obtains formulae for $\mathbf{F}_1$ and $\mathbf{G}_1$ which, when combined with equations (2.16) and (2.17), yield the values of $\mathbf{F}$ and $\mathbf{G}$ to $O(R)$. Hence

$$F_i = {}^1F_i + {}^2F_i + o(R), \tag{3.7}$$

where

$${}^1F_i = (A_{ij}V_j + D_{ij}\Omega_j) - \tfrac{3}{16}RA_{ij}\{3A_{jk}V_kV^2 + 3D_{jk}\Omega_kV^2 - (V_kA_{kl}V_l)V_j - (V_kD_{kl}\Omega_l)V_j\}/V, \tag{3.8a}$$

and

$${}^2F_i = (1/6\pi)R[K_{ilm}^{TRR}\Omega_l\Omega_m + 2K_{ilm}^{TRT}\Omega_lV_m + K_{ilm}^{TTT}V_lV_m] \tag{3.8b}$$

or alternatively, if $V \neq 0$,

$$^2F_i = (1/6\pi)R(\delta_{ip}-V_iV_p/V^2)[K^{TRR}_{plm}\Omega_l\Omega_m+2K^{TRT}_{plm}\Omega_lV_m+K^{TTT}_{plm}V_lV_m]$$
$$-(1/6\pi V^2)RV_i\Omega_p[K^{RRR}_{plm}\Omega_l\Omega_m+2K^{RRT}_{plm}\Omega_lV_m+K^{RTT}_{plm}V_lV_m], \quad (3.8c)$$

while similarly $$G_i = {}^1G_i+{}^2G_i+o(R), \quad (3.9)$$

where $$^1G_i = (B_{ij}V_j+E_{ij}\Omega_j)$$
$$-\tfrac{3}{16}RB_{ij}\{3A_{jk}V_kV^2+3D_{jk}\Omega_kV^2-(V_kA_{kl}V_l)V_j-(V_kD_{kl}\Omega_l)V_j\}/V \quad (3.10a)$$

and $$^2G_i = (1/6\pi)R[K^{RRR}_{ilm}\Omega_l\Omega_m+2K^{RRT}_{ilm}\Omega_lV_m+K^{RTT}_{ilm}V_lV_m], \quad (3.10b)$$

or alternatively, if $\Omega \neq 0$,

$$^2G_i = (1/6\pi)R(\delta_{ip}-\Omega_i\Omega_p/\Omega^2)[K^{RRR}_{plm}\Omega_l\Omega_m+2K^{RRT}_{plm}\Omega_lV_m+K^{RTT}_{plm}V_lV_m]$$
$$-(1/6\pi\Omega^2)RV_p\Omega_i[K^{TRR}_{plm}\Omega_l\Omega_m+2K^{TRT}_{plm}\Omega_lV_m+K^{TTT}_{plm}V_lV_m]. \quad (3.10c)$$

The various third-order tensors occurring in these relations may be written in terms of $\bar{u}_{ij}$ and $\hat{u}_{ij}$ as

$$\left.\begin{aligned}
K^{TRR}_{ilm} &= \int_\Gamma \bar{e}_{jki}\hat{u}_{jl}\hat{u}_{km}\,dV, \quad K^{TRT}_{ilm} = \int_\Gamma \bar{e}_{jki}\hat{u}_{jl}(\bar{u}_{km}-\delta_{km})\,dV,\\
K^{TTT}_{ilm} &= -\frac{1}{2}\left\{\int_{S_L}({}_{-2}\bar{u}_{li})\,dS_m+\int_{S_L}({}_{-2}\bar{u}_{mi})\,dS_l\right\}+\int_\Gamma \bar{e}_{jki}(\bar{u}_{jl}-\delta_{jl})(\bar{u}_{km}-\delta_{km})\,dV,\\
K^{RRR}_{ilm} &= \int_\Gamma \hat{e}_{jki}\hat{u}_{jl}\hat{u}_{km}\,dV, \quad K^{RRT}_{ilm} = \int_\Gamma \hat{e}_{jki}\hat{u}_{jl}(\bar{u}_{km}-\delta_{km})\,dV,\\
K^{RTT}_{ilm} &= -\frac{1}{2}\int_{S_L}\left\{({}_{-2}\hat{u}_{li})\,dS_m+\int_{S_L}({}_{-2}\hat{u}_{mi})\,dS_l\right\}+\int_\Gamma \hat{e}_{jki}(\bar{u}_{jl}-\delta_{jl})(\bar{u}_{km}-\delta_{km})\,dV,
\end{aligned}\right\} \quad (3.11)$$

in which the third-order tensor functions $\bar{e}_{jki}$ and $\hat{e}_{jki}$ are respectively defined to be the rate-of-strain tensors corresponding to the flows $\bar{u}_{ji}$ and $\hat{u}_{ji}$, i.e.

$$\bar{e}_{jki} = \tfrac{1}{2}(\bar{u}_{ji,k}+\bar{u}_{ki,j}), \quad \hat{e}_{jki} = \tfrac{1}{2}(\hat{u}_{ji,k}+\hat{u}_{ki,j}). \quad (3.12)$$

The quantities $({}_{-2}\bar{u}_{li})$ and $({}_{-2}\hat{u}_{li})$ are respectively the terms in the asymptotic expansions of $\bar{u}_{li}$ and $\hat{u}_{li}$ for large r which are homogeneous in r^{-2}. Thus the surface integrals in (3.11) which are taken over a large sphere S_L of radius L are in fact independent of the actual value of L.

4. Force and couple on body to $O(R^2 \ln R)$

The form of the inner expansion for $(\mathbf{u}, p)$ may to $O(R^2)$ be shown to take the form (see Brenner & Cox 1963)

$$\left.\begin{aligned}
\mathbf{u} &= \mathbf{u}_0+R\mathbf{u}_1+(R^2\ln R)\mathbf{u}_2+R^2\bar{\mathbf{u}}_2+o(R^2),\\
p &= p_0+Rp_1+(R^2\ln R)p_2+R^2\bar{p}_2+o(R^2),
\end{aligned}\right\} \quad (4.1)$$

where $(\bar{\mathbf{u}}_2, \bar{p}_2)$ satisfies the equations

$$\nabla^2\bar{\mathbf{u}}_2-\nabla\bar{p}_2 = \mathbf{u}_0.\nabla\mathbf{u}_1+\mathbf{u}_1.\nabla\mathbf{u}_0, \quad \nabla.\bar{\mathbf{u}}_2 = 0, \quad (4.2)$$

and $(\mathbf{u}_2, p_2)$ the Stokes equations

$$\nabla^2\mathbf{u}_2 - \nabla p_2 = \mathbf{0}, \quad \nabla.\mathbf{u}_2 = 0, \tag{4.3}$$

with the boundary conditions that

$$\mathbf{u}_2 = \mathbf{0} \quad \text{on} \quad B \qquad \text{and} \qquad \mathbf{u}_2 \to \mathbf{d} \quad \text{as} \quad r \to \infty, \tag{4.4}$$

where $\mathbf{d}$ is a constant vector determined by the term involving $\ln r$ in the asymptotic expansion of $\bar{\mathbf{u}}_2$ for large values of r. Such a term is of the form $\mathbf{d}\ln r$.

Thus by using the asymptotic expansions (2.11) and (2.14) for $\mathbf{u}_0$ and $\mathbf{u}_1$ one may evaluate the asymptotic expansion for $(\mathbf{u}_0.\nabla\mathbf{u}_1 + \mathbf{u}_1.\nabla\mathbf{u}_0)$ and hence find that term in the asymptotic expansion of $\mathbf{u}_2$ which involves $\ln r$ (see Cox 1964). Thus we obtain the value of $\mathbf{d}$ as

$$\mathbf{d} = \tfrac{3}{320}\{-31\mathbf{F}_0.\mathbf{V}\mathbf{F}_0 + 7F_0^2\mathbf{V}\}. \tag{4.5}$$

The force $\mathbf{F}$ and couple $\mathbf{G}$ acting on the body must be of the form

$$\left.\begin{aligned} \mathbf{F} &= \mathbf{F}_0 + R\mathbf{F}_1 + (R^2\ln R)\,\mathbf{F}_2 + O(R^2), \\ \mathbf{G} &= \mathbf{G}_0 + R\mathbf{G}_1 + (R^2\ln R)\,\mathbf{G}_2 + O(R^2), \end{aligned}\right\} \tag{4.6}$$

where $\mathbf{F}_2$ and $\mathbf{G}_2$ are the result of the flow field $(\mathbf{u}_2, p_2)$. As we have seen, this flow field is just the Stokes flow of the fluid past the body with velocity $\mathbf{d}$. Hence it follows that

$$(F_2)_i = -A_{ij}d_j, \quad (G_2)_i = -B_{ij}d_j. \tag{4.7}$$

By making use of the equations (4.5), (4.6), (4.7) and the values of $\mathbf{F}_0$, $\mathbf{G}_0$, $\mathbf{F}_1$ and $\mathbf{G}_1$ already obtained we may write down the value of $\mathbf{F}$ and $\mathbf{G}$ to $O(R^2\ln R)$ immediately as

$$F_i = {}^1F_i + {}^2F_i + O(R^2), \quad G_i = {}^1G_i + {}^2G_i + O(R^2), \tag{4.8}$$

where

$$\begin{aligned} {}^1F_i = {} & (A_{ij}V_j + D_{ij}\Omega_j) - \tfrac{3}{16}RA_{ij}\{3V^2(A_{jk}V_k + D_{jk}\Omega_k) \\ & - (V_kA_{kl}V_l + V_kD_{kl}\Omega_l)V_j\}/V \\ & + \tfrac{3}{320}(R^2\ln R)\{31(V_kA_{kl}V_l + V_kD_{kl}\Omega_l)A_{ij}(A_{jm}V_m + D_{jm}\Omega_m) \\ & - 7(A_{kl}V_l + D_{kl}\Omega_l)(A_{km}V_m + D_{km}\Omega_m)A_{ij}V_j\} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} {}^1G_i = {} & (B_{ij}V_j + E_{ij}\Omega_j) - \tfrac{3}{16}RB_{ij}\{3V^2(A_{jk}V_k + D_{jk}\Omega_k) \\ & - (V_kA_{kl}V_l + V_kD_{kl}\Omega_l)V_j\}/V \\ & + \tfrac{3}{320}(R^2\ln R)\{31(V_kA_{kl}V_l + V_kD_{kl}\Omega_l)B_{ij}(A_{jm}V_m + D_{jm}\Omega_m) \\ & - 7(A_{kl}V_l + D_{kl}\Omega_l)(A_{km}V_m + D_{km}\Omega_m)B_{ij}V_j\}, \end{aligned} \tag{4.10}$$

${}^2F_i$ and ${}^2G_i$ being given respectively by the equations (3.8*b*, *c*) and (3.10*b*, *c*).

5. General discussion

In the formulae (4.8), (4.9) and (4.10) for the force and couple on the body to $O(R^2\ln R)$, it should be noted that if we reverse the velocity $\mathbf{V}$ and angular velocity $\mathbf{\Omega}$ of the body, the force ${}^1\mathbf{F}$ and couple ${}^1\mathbf{G}$ are reversed, whereas ${}^2\mathbf{F}$ and ${}^2\mathbf{G}$ remain unaltered. It was to give ${}^1\mathbf{F}$, ${}^1\mathbf{G}$ and ${}^2\mathbf{F}$, ${}^2\mathbf{G}$ these properties that the force **F** and couple **G** were so divided into these parts. Whereas $^1\mathbf{F}$ and $^1\mathbf{G}$ may be calculated solely from a knowledge of the tensors A_{ij}, B_{ij}, D_{ij} and E_{ij} (i.e. the values of the forces and couples on the body due to the Stokes flows $\bar{u}_{ij}$ and $\hat{u}_{ij}$), it is necessary to know the complete fields $\bar{u}_{ij}$ and $\hat{u}_{ij}$ in order to calculate $^2\mathbf{F}$ and $^2\mathbf{G}$.

The force ^2F_i (for $\mathbf{V} \neq \mathbf{0}$) given by equation (3.8*c*) is seen to consist of the sum of the forces

$$+(1/6\pi)\,R(\delta_{ip}-V_iV_p/V^2)\,[K^{TRR}_{plm}\,\Omega_l\Omega_m+2K^{TRT}_{plm}\,\Omega_lV_m+K^{TTT}_{plm}\,V_lV_m]$$

and

$$-(1/6\pi)\,R(V_i\Omega_p/V^2)\,[K^{RRR}_{plm}\,\Omega_l\Omega_m+2K^{RRT}_{plm}\,\Omega_lV_m+K^{RTT}_{plm}\,V_lV_m],$$

the former being a *lift* force (perpendicular to V_i) and the latter a drag force (parallel to V_i) which vanishes for $\mathbf{\Omega} = \mathbf{0}$.

In a similar manner ^2G_i (for $\mathbf{\Omega} \neq \mathbf{0}$) given by equation (3.10*c*) consists of the sum of the couples

$$+(1/6\pi)\,R(\delta_{ip}-\Omega_i\Omega_p/\Omega^2)\,[K^{RRR}_{plm}\,\Omega_l\Omega_m+2K^{RRT}_{plm}\,\Omega_lV_m+K^{RTT}_{plm}\,V_lV_m]$$

and

$$-(1/6\pi)\,R(\Omega_iV_p/\Omega^2)\,[K^{TRR}_{plm}\,\Omega_l\Omega_m+2K^{TRT}_{plm}\,\Omega_lV_m+K^{TTT}_{plm}\,V_lV_m],$$

the former being perpendicular to $\mathbf{\Omega}$ and the latter parallel to $\mathbf{\Omega}$ and vanishing for $\mathbf{V} = \mathbf{0}$.

Should the body B possess certain types of symmetry properties, then restrictions are automatically imposed upon the form which may be taken by the various tensors occurring in the equations (3.8) and (3.10) for the force and couple on the body (see Brenner & Cox 1963) and Cox (1964). The form the restrictions take for a particular tensor and for a given body symmetry property depends on the order of the tensor and whether it is a real tensor (i.e. a relative tensor of *even* weight) or a pseudotensor (a relative tensor of *odd* weight). We note that in the relations (3.8) and (3.10) the types of the tensors occurring are:

(i) second-order real tensors, these being A_{ij} and E_{ij} for which there exist the relations

$$A_{ij} = A_{ji}, \quad E_{ij} = E_{ji}; \tag{5.1}$$

(ii) second-order pseudotensors, these being B_{ij} and D_{ij};

(iii) third-order real tensors, these being K^{TRR}_{plm}, K^{TTT}_{plm}, K^{RRT}_{plm} for which

$$K^{TRR}_{plm} = K^{TRR}_{pml}, \quad K^{TTT}_{plm} = K^{TTT}_{pml}; \tag{5.2}$$

(iv) third-order pseudotensors, these being K^{TRT}_{plm}, K^{RRR}_{plm}, K^{RTT}_{plm} for which

$$K^{RRR}_{plm} = K^{RRR}_{pml}, \quad K^{TTT}_{plm} = K^{TTT}_{pml}. \tag{5.3}$$

Hence, for bodies possessing symmetry properties one may write the equations (3.8) and (3.10) for **F** and **G** in a simplified form. In particular one finds that $^2\mathbf{F}$ or $^2\mathbf{G}$ (or certain of their components) vanish for certain types of symmetric bodies thus enabling the calculation of **F** and **G** (or certain of their components) from the values of the tensors A_{ij}, B_{ij}, D_{ij}, E_{ij} alone, the calculation of $\bar{u}_{ij}$ and $\hat{u}_{ij}$ and of the various third-order tensors K^{TRR}_{plm}, K^{TTT}_{plm}, etc., then being not required.

6. Two classes of problems

We now examine the forms taken by the formulae (3.8) and (3.10) for the two classes of problems described in § 2. The first of these (class (*a*)) is that of a body of arbitrary shape in pure translation whilst the second (class (*b*)) is that of an axially symmetric body in translation in any direction together with a rotation about its symmetry axis.

Class (a)

For non-rotating bodies, the formula for the force $\mathbf{F}$ takes the form

$$F_i = {}^1F_i + {}^2F_i + O(R^2), \tag{6.1a}$$

where
$$\begin{aligned}{}^1F_i = {} & A_{ij}V_j - (3/16V)R\{3V^2A_{ij} - \delta_{ij}(V_kA_{kl}V_l)\}A_{jm}V_m \\ & + \tfrac{3}{320}(R^2\ln R)\{31A_{ij}(V_kA_{kl}V_l) - 7\delta_{ij}(V_lA_{kl}A_{km}V_m)\}A_{jn}V_n,\end{aligned} \tag{6.1b}$$

and
$${}^2F_i = (1/6\pi)R(\delta_{ip} - V_iV_p/V^2)V_lV_mK^{TTT}_{plm}, \tag{6.1c}$$

whilst that for the couple $\mathbf{G}$ takes the form

$$G_i = {}^1G_i + {}^2G_i + O(R^2), \tag{6.2a}$$

where
$$\begin{aligned}{}^1G_i = {} & B_{ij}V_j - (3/16V)RB_{ij}\{3V^2A_{jk}V_k - (V_kA_{kl}V_l)V_j\} \\ & + \tfrac{3}{320}(R^2\ln R)B_{ij}\{31(V_kA_{kl}V_l)A_{jm}V_m - 7(V_lA_{kl}A_{km}V_m)V_j\}\end{aligned} \tag{6.2b}$$

and
$${}^2G_i = (1/6\pi)RV_lV_mK^{RTT}_{ilm}, \tag{6.2c}$$

the tensors K^{TTT}_{plm}, K^{RTT}_{ilm} being given by the equations (3.11).

The value of $\mathbf{F}$ so obtained is in complete agreement with that obtained by Brenner & Cox (1963) and all the results given there regarding the vanishing of ${}^2\mathbf{F}$ (or certain of its components) for symmetric bodies still hold. However, in addition we can now set up, by the use of the fact that K^{RTT}_{ilm} is a third-order pseudotensor, a list of body symmetry conditions implying the vanishing of ${}^2\mathbf{G}$ (or certain of its components); e.g. (i) ${}^2\mathbf{G} = 0$ for all $\mathbf{V}$ if the body transforms into itself under rotations of $\frac{1}{2}\pi$ about axes 1 and 2, (ii) ${}^2G_2 = {}^2G_3 = 0$ for $\mathbf{V}$ lying in plane 1 if the body transforms into itself under *either* a rotation of π about axis 1, *or* a reflexion in plane 1. A more complete list is given by Cox (1964).

Class (b)

We consider now a body axially symmetric about an axis 1 which rotates with an angular velocity $\mathbf{\Omega} = (\Omega_1, 0, 0)$ and translates with a velocity $\mathbf{V}$ which, by a suitable choice of axes, may be written as $\mathbf{V} = (V_1, V_2, 0)$. From the equations (3.8) and by the forms taken by the various tensors for axially symmetric bodies it may be shown that the force $\mathbf{F}$ on the body is the same as that for a translating *non-rotating* body except that there is now an *additional* force $\mathbf{F}^*$ given by

$$\left.\begin{aligned}F^*_1 &= (1/6\pi)R\{K^{TRR}_{111}(1 - V_1^2/V^2)\Omega_1^2 + 2K^{RTT}_{111}V_1^2\Omega_1^2/V^2\},\\ F^*_2 &= (1/6\pi)R\{-K^{TRR}_{111}\Omega_1^2V_1V_2/V^2 - 2K^{RRT}_{111}\Omega_1^2V_1V_2/V^2\},\\ F^*_3 &= (1/6\pi)R\{2K^{TRT}_{312}\Omega_1V_2\}.\end{aligned}\right\} \tag{6.3}$$

In a similar manner, it may be shown that the couple **G** is the same as that for the non-rotating body with an additional couple **G*** given by

$$\left.\begin{aligned} G_1^* &= E_{11}\Omega_1 + (1/6\pi)R\{-K_{111}^{TRR}\Omega_1 V_1\},\\ G_2^* &= (1/6\pi)R\{+2K_{212}^{RTT}\Omega_1 V_2\},\\ G_3^* &= 0, \end{aligned}\right\} \tag{6.4}$$

the term $E_{11}\Omega_1$ occurring in G_1^* being the Stokes rotational drag.

For an axially symmetric body *with fore-aft symmetry*, the extra force **F*** and couple **G*** due to the rotation, reduce to

$$F_1^* = F_2^* = 0, \quad F_3^* = (1/6\pi)R\{2K_{312}^{TRT}\Omega_1 V_2\} \tag{6.5}$$

and

$$G_1^* = E_{11}\Omega_1, \quad G_2^* = G_3^* = 0. \tag{6.6}$$

Thus the effect of rotation on such a body is to produce a lift force on the body in the direction of axis 3. Rubinow & Keller (1961) obtained the value of this force for the particular case of the body being a sphere, its velocity and angular velocity being respectively $\mathbf{V} = (0, V_2, 0)$ and $\mathbf{\Omega} = (\Omega_1, 0, 0)$. This problem will be considered in greater detail in § 8.

A rotating axially symmetric body (without fore-aft symmetry) which is *not* translating relative to the fluid at infinity† possesses a force and couple acting upon it, their *complete* values (see equations (6.3) and (6.4)) being given by

$$F_1 = (1/6\pi)RK_{111}^{TRR}\Omega_1^2 + O(R^2), \quad F_2 = F_3 = 0 \tag{6.7}$$

and

$$G_1 = E_{11}\Omega_1 + O(R^2), \quad G_2 = G_3 = 0, \tag{6.8}$$

the axis of the body and of the rotation being taken as the axis 1. Thus there exists a force proportional to $R\Omega_1^2$ acting on the body in the direction of its axis which is unaltered by a reversal of the angular velocity. The couple on the body to the order in R considered is just the Stokes couple for the given rotation.

7. Two examples of class (*a*) problems

(i) *Example* 1: *A spheroid in pure translation*

For a centrally symmetric body in pure translation the *complete* force **F** and couple **G** acting upon it are

$$\left.\begin{aligned} F_i &= A_{ij}V_j - (3/16V)R\{3V^2A_{ij} - \delta_{ij}(V_kA_{kl}V_l)\}A_{jm}V_m\\ &\quad + \tfrac{3}{320}(R^2\ln R)\{31A_{ij}(V_kA_{kl}V_l) - 7\delta_{ij}(V_lA_{kl}A_{km}V_m)\}A_{jn}V_n + O(R^2),\\ G_i &= (1/6\pi)RK_{ilm}^{RTT}V_lV_m + O(R^2). \end{aligned}\right\} \tag{7.1}$$

For such a body, we note that whereas the Stokes couple is identically zero, there is in general a couple of $O(R)$ acting on the body. This we shall calculate for a body whose surface is that of a spheroid of small eccentricity. Using

† For a body not translating relative to a fluid at rest at infinity, the velocity in the outer region of expansion is of the form

$$\mathbf{u} = R\mathbf{U}_1 + o(R),$$

where $\mathbf{U}_1$ is the *Stokes* velocity due to a point force. Hence the value of **k** is zero. The equations (4.8) to (4.10) may therefore be used without modification when $\mathbf{V} = \mathbf{0}$.

rectangular Cartesian co-ordinates (r_1, r_2, r_3) with origin fixed in the body, we assume the body surface B to have the (dimensionless) form

$$r = 1+\epsilon\left\{r^3\frac{\partial^2}{\partial r_1^2}\left(\frac{1}{r}\right)\right\}, \tag{7.2}$$

the term $r^3\,\partial^2(1/r)/\partial r_1^2$ being a surface harmonic of order two. The parameter ϵ is assumed so small that squares and higher-order terms may be neglected. Such a surface is a spheroid axially symmetric about axis 1.

We may now choose our axes such that the velocity $\mathbf{V}$ of the body is $(V\cos\alpha,\ V\sin\alpha,\ 0)$ α then being the angle between the translation velocity and the axis of symmetry. It may then be shown from equations (6.2) that the couple $\mathbf{G}$ on the body is given by

$$G_1 = G_2 = O(R^2), \quad G_3 = (1/3\pi)\,RK_{312}^{RTT}\,V^2\sin\alpha\cos\alpha+O(R^2). \tag{7.3}$$

G_1 and G_2 must in fact be zero to all orders in R by symmetry.

By making use of the results obtained by Brenner (1964a) for the Stokes flows due to a slightly deformed sphere in translation or rotation we may obtain the values of $\bar{u}_{ij}$ and $\hat{u}_{ij}$ for a body whose surface is that described by equation (7.2). These, when substituted into the equation for K_{312}^{RTT} (see equations (3.11)), yield after a long and tedious calculation (see Cox 1964)

$$K_{312}^{RTT} = -\frac{87\pi}{40}\epsilon+O(\epsilon^2), \tag{7.4}$$

where all terms of $O(\epsilon^2)$ have been neglected and use has been made of the orthogonality properties of spherical harmonics. Hence the couple on the body is

$$G_1 = 0, \quad G_2 = 0, \quad G_3 = -\tfrac{29}{40}R\epsilon V^2\sin\alpha\cos\alpha+O(R^2). \tag{7.5}$$

Thus we see that in general there does exist a couple of $O(R)$ on a centrally symmetric body in pure translation.

In § 9 we shall consider the free fall of a body with surface B given by (7.2). For such a motion it will be shown that the equations (7.5) may be used, although the fluid motion is then not steady.

(ii) *Example 2: A dumb-bell-shaped body in translation*

We consider here an example to show that for an arbitrary body in pure translation, the force $\mathbf{F}$ acting upon it, given by equation (6.1a), is such that it may have a component of *lift* which is not reversed by a reversal of $\mathbf{V}$ (i.e. that ${}^2\mathbf{F}$ is non-zero). An attempt was made to prove this result in Appendix 2 of a paper by Brenner & Cox (1963) in which an error occurred.† However, the general result is correct as will be shown by the following example.

Consider a dumb-bell-shaped body defined in dimensional variables as consisting of two centrally symmetric bodies P and Q whose dimensions are of the

† This error occurred on page 593, line 3, which should have read

$$H(a_1 a_2 \dots a_u) = r^{-2u-1}\text{ (homogeneous polynomial of degree } u \text{ in } r_i).$$

The argument following this statement is therefore rendered invalid.

same order of magnitude a and are connected by a very thin rigid rod of width of order d and length b, so as to form one complete composite body. It is assumed that

$$d \ll a \ll b, \tag{7.6}$$

so that one may neglect the effect of the joining rod and assume that the mutual effect of the two bodies P and Q is small. Then it may be shown that for such a body the values of the components of the tensors K^{TRR}_{plm}, K^{TTT}_{plm} and K^{RRT}_{plm} may be calculated and hence the value of ${}^2\mathbf{F}$ (see Cox 1964).

Choosing rectangular Cartesian axes with the O_1 axis lying along PQ, we restrict ourselves to bodies in which the component bodies P and Q are ellipsoids of revolution with their axes of symmetry lying in the plane 3 (i.e. the plane containing axes 1 and 2) in such a manner that the body Q is the mirror image of P in a plane which is the perpendicular bisector of the line PQ. It may then be shown (Cox 1964) that the force $\mathbf{F}$ (made dimensionless by the length b) acting on such a body is

$$\mathbf{F} = {}^1\mathbf{F} + {}^2\mathbf{F} + O(R^2),$$

where ${}^1\mathbf{F}$ is given by

$$\left.\begin{aligned} {}^1F_1 &= A_{11}V - \tfrac{3}{8}R(A_{11})^2V^2 + \tfrac{9}{40}(R^2\ln R)(A_{11})^3V^3, \\ {}^1F_2 &= 0, \quad {}^1F_3 = 0 \end{aligned}\right\} \tag{7.7}$$

and ${}^2\mathbf{F}$ by

$$\left.\begin{aligned} {}^2F_1 &= 0, \quad {}^2F_3 = 0, \\ {}^2F_2 &= \tfrac{3}{8}(a/b)^2RV^2A^P_{12}(3A^P_{22} - 2A^P_{11}), \end{aligned}\right\} \tag{7.8}$$

where A^P_{ij} is the value of the Stokes resistance tensor (made dimensionless by the length a) for the body P alone. The quantity A_{11} occurring in (7.7) is

$$A_{11} = 2(a/b)A^P_{11} + O(a/b)^2. \tag{7.9}$$

Thus the value of A^P_{ij} may be chosen such that ${}^2F_2$ is non-zero, thus giving us our desired result. It should also be noted that in the above example ${}^2\mathbf{F}$ is non-zero despite the fact that ${}^1\mathbf{F}$ is a pure *drag* force to $O(R^2\ln R)$, the velocity $\mathbf{V}$ being in the direction of a Stokes principal axis of resistance (i.e. in the direction of a principal axis of the tensor A_{ij}). Thus we have shown that even for translation in the direction of a principal axis of resistance we may have a lift force on a body of $O(R)$ which is unaltered by a reversal of the direction of translation.

8. Two examples of class (*b*) problems

(i) *Example* 1: *A rotating translating sphere*

Consider a sphere being translated with velocity $\mathbf{V} = (V\cos\theta, V\sin\theta, 0)$ and constrained to rotate with angular velocity $\mathbf{\Omega} = (\Omega, 0, 0)$. Then by the use of equations (6.5) and (6.6), it may be seen that the *complete* force and couple on the sphere are given by

$$\left.\begin{aligned} F_1 &= V\cos\theta\{A_{11} - \tfrac{3}{8}RA^2_{11} + \tfrac{9}{40}(R^2\ln R)A^3_{11}\} + O(R^2), \\ F_2 &= V\sin\theta\{A_{11} - \tfrac{3}{8}RA^2_{11} + \tfrac{9}{40}(R^2\ln R)A^3_{11}\} + O(R^2), \\ F_3 &= (1/6\pi)R\{2K^{TRT}_{312}\Omega V\sin\theta\} + O(R^2), \end{aligned}\right\} \tag{8.1a}$$

and

$$G_1 = E_{11}\Omega + O(R^2), \quad G_2 = G_3 = O(R^2), \tag{8.1b}$$

use having been made of the form taken by the tensors A_{ij}, B_{ij}, etc., for a sphere.

The value of K_{312}^{TRT} may be found from the relations (3.11), the tensor function $\bar{u}_{ij}$ and $\hat{u}_{ij}$ being easily obtainable for a sphere. Also, since $A_{11} = -1$ and $E_{11} = -\frac{4}{3}$, we obtain finally

$$\left.\begin{aligned} F_1 &= -V\cos\theta(1+\tfrac{3}{8}R+\tfrac{9}{40}R^2\ln R)+O(R^2),\\ F_2 &= -V\sin\theta(1+\tfrac{3}{8}R+\tfrac{9}{40}R^2\ln R)+O(R^2),\\ F_3 &= +\tfrac{1}{8}R\Omega V\sin\theta+O(R^2),\end{aligned}\right\} \tag{8.2a}$$

and

$$G_1 = -\tfrac{4}{3}\Omega+O(R^2), \quad G_2 = G_3 = O(R^2). \tag{8.2b}$$

When the velocity of particle translation and its axis of rotation are perpendicular (i.e. $\theta = \frac{1}{2}\pi$) the above values for $\mathbf{F}$ and $\mathbf{G}$ are in complete agreement with Rubinow & Keller's (1961) results for this case.

(ii) *Example 2: a dumb-bell-shaped body in pure rotation*

We consider now an axially symmetric body (without fore-aft symmetry) consisting of a dumb-bell as defined in § 7 (example 2) in which the component bodies P and Q are spheres of radii $\lambda^P a$ and $\lambda^Q a$ respectively, λ^P and λ^Q being unequal and of order unity in magnitude. Suppose such a body is rotating about its axis of symmetry with angular velocity Ω, the velocity of translation $\mathbf{V}$ being zero. Then, if we take axes as before with the axis 1 lying along PQ, the force $\mathbf{F}$ and couple $\mathbf{G}$ on the body are given by the equations (6.7) and (6.8). The value of the tensor component K_{111}^{TRR} may be shown to be (Cox 1964)

$$K_{111}^{TRR} = 2\pi\lambda^P\lambda^Q\{(\lambda^Q)^4-(\lambda^P)^4\}(a/b)^6+O(a/b)^7.$$

Hence

$$F_1 = \tfrac{1}{3}\lambda^P\lambda^Q\{(\lambda^Q)^4-(\lambda^P)^4\}(a/b)^6R\Omega^2+O(R^2), \quad F_2 = F_3 = 0.$$

Thus there exists a force on the body in the direction of its axis of symmetry from the smaller to the larger sphere. It should be noted that this force is proportional to Ω^2 and therefore acts in the same direction along the axis, whatever the direction of body rotation.

9. Quasi-steady problems

The equations obtained in § 4 for the force and couple on a body to order $(R^2 \ln R)$ have been obtained on the assumption that the motion is steady relative to some uniformly translating frame of reference. However, the theory would be expected to remain valid for non-steady motions so long as they are such that their time scales of the variations of the velocity $\mathbf{V}$ and body orientation are very large, i.e. in our dimensionless variables $d\mathbf{V}/dt$ and $\mathbf{\Omega}$ are very much less than unity. Such motions we shall call quasi-steady.

Consider a body moving through a fluid with a given external (dimensionless) force $\mathbf{F}^*(=\mathbf{F}^{*\prime}/6\pi\mu c U)$ and couple $\mathbf{G}^*(=\mathbf{G}^{*\prime}/6\pi\mu c^2 U)$ acting upon it (about an origin fixed in the body at its centre of mass). The equations of motion for the body then take the dimensionless forms

$$\left.\begin{aligned} 6\pi(F_i+F_i^*) &= (\rho_s/\rho)RM\,dV_i/dt,\\ 6\pi(G_i+G_i^*) &= (\rho_s/\rho)R\,d(I_{ij}\Omega_j)/dt,\end{aligned}\right\} \tag{9.1}$$

where ρ_s is a characteristic density of the body, M is the dimensionless mass of the body $(=M'/\rho_s c^3)$, I_{ij} is the dimensionless inertia tensor of the body

($= I'_{ij}/\rho_s c^5$) about the origin and F_i and G_i are the dimensionless force and couple acting on the body about the origin as a result of the fluid motion.

Under conditions of equilibrium orientation (i.e. for $\mathbf{\Omega} = 0$ and $\mathbf{V}$ independent of time) the equations of motion (9.1) become

$$F_i + F_i^* = 0, \quad G_i + G_i^* = 0, \tag{9.2}$$

where F_i and G_i are given to $O(R^2 \ln R)$ in § 4, since under such conditions the fluid motion is steady. However, the full equations (9.1) may be used in conjunction with the values of F_i and G_i given in § 4 for motions which are arbitrarily near an equilibrium state since for such motions $d\mathbf{V}/dt$ and $\mathbf{\Omega}$ may be made arbitrarily small (of order δ, say), the effect of non-steadiness of fluid motion then being of order $R\delta$. (Note: care must be taken to include all terms to whatever order one is working.)

There are circumstances in which the equation (9.1) may be used in conjunction with the values of $\mathbf{F}$ and $\mathbf{G}$ given as in § 4, to give the complete body motion. For example, consider a body which is undergoing a motion in which dV_i/dt and Ω_i are both of order δ where δ is a small parameter. Then if R is small and ρ_s/ρ so large that $(R\rho_s/\rho)$ is of order unity (e.g. a solid particle falling through a gas), we may use the expression in § 4 for F_i and G_i in conjunction with (9.1) so long as all terms have been included to the order to which we are working.

As an example of a body exhibiting such quasi-steady motion we consider the free fall in a fluid of a spheroidal-shaped body of uniform density whose surface is described by equation (7.2). By taking the origin of co-ordinates at the centre of such a body, the equations of motion (9.1) yield

$$6\pi F_i^* + 6\pi A_{ij} V_j - (3/16\,V)\,R\,.\,6\pi\{3V^2 A_{ij} - \delta_{ij}(V_k A_{kl} V_l)\} A_{jm} V_m + 2R\{K_{ilm}^{TRT}\,\Omega_l V_m\} = (\rho_s/\rho) RM\, dV_i/dt, \tag{9.3}$$

$$6\pi E_{ij}\,\Omega_j + R\{K_{ilm}^{RRR}\,\Omega_l \Omega_m + K_{ilm}^{RTT}\, V_l V_m\} = (\rho_s/\rho) R\, d(I_{ij}\,\Omega_j)/dt, \tag{9.4}$$

where we have neglected the terms in $(R^2 \ln R)$ in the formulae for F_i and G_i. It will now be shown that for such a motion that $\mathbf{\Omega}$ and $d\mathbf{V}/dt$ are small thus verifying the assumption that the effects of non-steadiness may be neglected.

We suppose that ρ_s/ρ is large such that $(\rho_s/\rho R)$ is of order unity. Assume that $\mathbf{\Omega} = O(\delta)$, where δ is a small parameter. The time required for the body to change its orientation by an amount of order unity will then be $O(\delta^{-1})$. Thus $d\mathbf{\Omega}/dt = O(\delta^2)$ and $d\mathbf{V}/dt = O(\delta)$. Hence, in equation (9.4) the term $E_{ij}\,\Omega_j$ is of order δ, the term $RK_{ilm}^{RRR}\,\Omega_l\Omega_m$ is of order $R\delta^2$, the term $RK_{ilm}^{RTT}\, V_l V_m$ is of order R, and the term $(\rho_s/\rho) R\, d(I_{ij}\,\Omega_j)/dt$ is of order δ^2. Therefore if this equation is to be satisfied δ must be taken equal to R. Thus the time scale for the change in body orientation is R^{-1}. We shall therefore write

$$t = R^{-1}\tilde{t}, \tag{9.5}$$

and expand $\mathbf{V}$ and $\mathbf{\Omega}$ in terms of R as

$$\left.\begin{aligned} V_i &= (V_0)_i + R(V_1)_i + o(R), \\ \Omega_i &= R(\Omega_1)_i + o(R). \end{aligned}\right\} \tag{9.6}$$

Substituting (9.5) and (9.6) into the equations (9.3) and (9.4) and equating like powers in R, we obtain

$$F_i^* + A_{ij}(V_0)_j = 0, \tag{9.7}$$

$$6\pi A_{ij}(V_1)_j - (3/16V_0)\, 6\pi\{3(V_0)^2 A_{ij} - \delta_{ij}[(V_0)_k A_{kl}(V_0)_l]\} A_{jm}(V_0)_m = (\rho_s/\rho) RM\, d(V_0)_i/dt, \tag{9.8}$$

$$6\pi E_{ij}(\Omega_1)_j + K_{ilm}^{RTT}(V_0)_l(V_0)_m = 0, \tag{9.9}$$

the effect of the unsteadiness of the motion being of order R^2. Equation (9.7) gives the velocity $\mathbf{V}_0$ in terms of the body orientation, which when substituted into (9.9) gives the angular velocity (Ω_1). This in turn determines the change in body orientation, thus giving $\mathbf{\Omega}_1$ and $\mathbf{V}_0$ as functions of time. The value of $\mathbf{V}_0$, when substituted into (9.8), gives $\mathbf{V}_1$ as a function of time.

Take axes fixed in our spheroidal body such that the axis 1 lies along the symmetry axis and the force vector $\mathbf{F}^*$ lies in the plane 3 (containing axes 1 and 2), the assumption here being made that, throughout the body motion, the symmetry axis always remains in the same vertical plane. We let α be the angle between the axis 1 and the gravitational force $\mathbf{F}^*$. Since, for our body,

$$A_{ij} = -\delta_{ij} + O(\epsilon), \tag{9.10}$$

it follows from (9.7) that the angle between $\mathbf{F}^*$ and $\mathbf{V}_0$ is of order ϵ. Hence using the value of the tensor K_{ilm}^{RTT} obtained in § 7 the equation (9.9) yields

$$6\pi E_{ij}(\Omega_1)_j + \delta_{i3}(-\tfrac{87}{20}\pi\epsilon)\, V^2 \sin\alpha\cos\alpha = 0, \tag{9.11}$$

where terms of $O(\epsilon^2)$ have been neglected.

For the spheroid

$$E_{ij} = -\tfrac{4}{3}\delta_{ij} + O(\epsilon), \tag{9.12}$$

from which it may be deduced that the angular velocity $\mathbf{\Omega} = R\mathbf{\Omega}_1 + o(R)$ of the body is of the form $(0, 0, \Omega_3)$, where

$$\Omega_3 = -\tfrac{87}{160} R\epsilon V^2 \sin\alpha\cos\alpha + o(R).$$

Since $\Omega_3 = -d\alpha/dt$, it follows that we now have an ordinary differential equation for α which may be solved to give

$$\tan\alpha = A\exp(\tfrac{87}{160} R\epsilon V^2 t), \tag{9.14}$$

where A is an arbitrary constant determined by the value of α at $t = 0$.

Thus there are two positions of equilibrium orientation, one for which the body axis is vertical ($\alpha = 0$) and one for which it is horizontal ($\alpha = \frac{1}{2}\pi$). The equilibrium orientation for which $\alpha = 0$ is stable for $\epsilon < 0$ and unstable for $\epsilon > 0$, whereas that for $\alpha = \frac{1}{2}\pi$ is unstable for $\epsilon < 0$ and stable for $\epsilon > 0$. Hence a body whose shape is that of a spheroid of small eccentricity would, upon falling through a viscous fluid, take up a position with its axis horizontal if it is prolate (i.e. if $\epsilon > 0$), or a position with its axis vertical if oblate.

Brenner (1964a) has shown that the resistance tensor A_{ij} for this particular example is given by

$$A_{11} = -1 + \tfrac{2}{5}\epsilon + O(\epsilon^2), \quad A_{22} = A_{33} = -1 - \tfrac{1}{5}\epsilon + O(\epsilon^2), \tag{9.15}$$

all other A_{ij} being zero. Thus the body takes up an orientation which makes its resistance to motion a maximum (i.e. an orientation which makes its velocity a minimum).

It is seen that the equation (9.14) is unaltered if (ρ_s/ρ) is of order unity, the only difference now being that the term on the right-hand side of equation (9.8) may be omitted.

10. Oseen force and couple on body to $O(R)$

For the case of the sphere (Proudman & Pearson 1957) and spheroid (Breach 1961), the value of the force to $O(R)$ as calculated by the singular perturbation technique is identical with that predicted by the classical Oseen equations, despite the fact that the latter do not furnish the correct asymptotic behaviour to $O(R)$ in the velocity field. We therefore consider the Oseen equations for a non-rotating body which may be written in the dimensionless form

$$\nabla^2\mathbf{u} - \nabla p = -R\mathbf{V}.\nabla\mathbf{u}, \quad \nabla.\mathbf{u} = 0 \tag{10.1}$$

with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad B, \qquad \mathbf{u} \to -\mathbf{V} \quad \text{as} \quad r \to \infty. \tag{10.2}$$

The force $\overline{\mathbf{F}}$ and couple $\overline{\mathbf{G}}$, say, acting on the body due to such a velocity field may be calculated to $O(R)$, in a manner similar to that of § 3. Hence we obtain (see Cox 1964)

$$\overline{F}_i = {}^1\overline{F}_i + {}^2\overline{F}_i + o(R), \tag{10.3}$$

where

$$\left.\begin{aligned} {}^1\overline{F}_i &= A_{ij}V_j - (3/16V)RA_{ij}\{3A_{jk}V_k V^2 - (V_k A_{kl} V_l)V_j\}, \\ {}^2\overline{F}_i &= (1/6\pi)R(\delta_{ip} - V_i V_p/V^2)\{\overline{K}^{TTT}_{plm}V_l V_m\}, \end{aligned}\right\} \tag{10.4}$$

and

$$\overline{G}_i = {}^1\overline{G}_i + {}^2\overline{G}_i + o(R) \tag{10.5}$$

where

$$\left.\begin{aligned} {}^1\overline{G}_i &= B_{ij}V_j - (3/16V)RB_{ij}\{3A_{jk}V_k V^2 - (V_k A_{kl} V_l)V_j\}, \\ {}^2\overline{G}_i &= (1/6\pi)R\{\overline{K}^{RTT}_{ilm}V_l V_m\}, \end{aligned}\right\} \tag{10.6}$$

the third-order tensors $\overline{K}^{TTT}_{plm}$ and $\overline{K}^{RTT}_{plm}$ being given by

$$\overline{K}^{TTT}_{ilm} = -\frac{1}{2}\left\{\int_{S_L}({}_{-2}\overline{u}_{li})\,dS_m + \int_{S_L}({}_{-2}\overline{u}_{mi})\,dS_l\right\} - \frac{1}{2}\left[\int_\Gamma \overline{u}_{ki,l}\{\overline{u}_{km} - \delta_{km}\}\,dV + \int_\Gamma \overline{u}_{ki,m}\{\overline{u}_{kl} - \delta_{kl}\}\,dV\right] \tag{10.7}$$

and

$$\overline{K}^{RTT}_{ilm} = -\frac{1}{2}\left\{\int_{S_L}({}_{-2}\hat{u}_{li})\,dS_m + \int_{S_L}({}_{-2}\hat{u}_{mi})\,dS_l\right\} - \frac{1}{2}\left[\int_\Gamma \hat{u}_{ki,l}\{\overline{u}_{km} - \delta_{km}\}\,dV + \int_\Gamma \hat{u}_{ki,m}\{\overline{u}_{kl} - \delta_{kl}\}\,dV\right]. \tag{10.8}$$

These solutions should be compared with the results given in § 3 for the full Navier–Stokes equations. Thus it is observed that for non-rotating bodies the formulae for the force and couple to $O(R)$, as calculated from the Oseen equations, is identical with that obtained from the full Navier–Stokes equations except that the third-order tensors K^{TTT}_{ilm} and K^{RTT}_{ilm} (given by equations (3.11)) are replaced by different tensors $\bar{K}^{TTT}_{ilm}$ and $\bar{K}^{RTT}_{ilm}$. It should also be noted that the force on the body as obtained from the Oseen equations and given by (10.3), (10.4) and (10.7) is the same as that given by Brenner & Cox (1963). For bodies with the symmetry properties for which $K^{TTT}_{ilm} = 0$, one may conclude also that $\bar{K}^{TTT}_{ilm} = 0$ (see Brenner & Cox 1963), showing that for such bodies (e.g. a sphere or a spheroid) one may conclude that

$$\bar{\mathbf{F}} = \mathbf{F} + o(R). \tag{10.9}$$

Similarly for certain other types of body symmetry for which $K^{RTT}_{ilm} = 0$, we have $\bar{K}^{RTT}_{ilm} = 0$ (see Cox 1964), showing that for such bodies

$$\bar{\mathbf{G}} = \mathbf{G} + o(R). \tag{10.10}$$

However, even for non-rotating bodies considered in this section, we might have the forces $\bar{\mathbf{F}}$ and $\mathbf{F}$ not equal to $O(R)$, i.e. the force on the body as calculated by the Oseen equations incorrect to $O(R)$. An attempt was made (Brenner & Cox 1963, Appendix 2) to give an example of a body for which $\bar{\mathbf{F}} \neq \mathbf{F}$ by the consideration of a body whose shape was a slightly deformed sphere. However an error was made which we will now rectify by considering a dumb-bell-shaped body for which it will be shown that the lift force on it is incorrect to $O(R)$ as calculated by the use of the Oseen equations.

Thus we again consider the dumb-bell-shaped body defined in § 7 (example 2) in which the component bodies P and Q are ellipsoids of revolution with their axes of symmetry lying in the plane 3 (PQ defining the axis 1). However we do *not* take this time the body Q to be the mirror image of P in the perpendicular bisecting plane of PQ. Thus if we take

$$\mathbf{\Omega} = \mathbf{0} \quad \text{and} \quad \mathbf{V} = (V, 0, 0), \tag{10.11}$$

then, by the symmetry of the body, it is seen that the value of $(\mathbf{F} - \bar{\mathbf{F}})$ correct to $O(R)$ is given by

$$F_1 - \bar{F}_1 = 0, \quad F_3 - \bar{F}_3 = 0, \quad F_2 - \bar{F}_2 = (1/6\pi) R V^2 (K^{TTT}_{211} - \bar{K}^{TTT}_{211}). \tag{10.12}$$

By using the values of the tensors K^{TTT}_{ilm} and $\bar{K}^{TTT}_{ilm}$ given in the equations (3.11) and (10.7) it may be shown after a long calculation that the shapes and orientations of the component bodies P and Q may be chosen so as to make the quantity $(K^{TTT}_{211} - \bar{K}^{TTT}_{211})$ non-zero (see Cox 1964). Thus the force $\bar{\mathbf{F}}$ on the body as calculated by the classical Oseen method is to $O(R)$ different from the force $\mathbf{F}$ as calculated by the singular perturbation method. Therefore it may be concluded that, although the classical Oseen method gives the correct *drag* to $O(R)$, it does not in general furnish the correct value for the *lift* force.

The author is grateful to the Department of Scientific and Industrial Research for a Research Studentship. Thanks are also due to Dr J. R. A. Pearson, University of Cambridge, and to Prof. H. Brenner of New York University for their interest and encouragement.

Part of this work was supported by a grant from the National Science Foundation (Grant no. GK-56).

REFERENCES

Breach, D. R. 1961 *J. Fluid Mech.* **10**, 306.

Brenner, H. 1963 *Chem. Engng Sci.* **18**, 1.

Brenner, H. 1964*a* *Chem. Engng Sci.* **19**, 519.

Brenner, H. 1964*b* *Chem. Engng Sci.* **19**, 599.

Brenner, H. & Cox, R. G. 1963 *J. Fluid Mech.* **17**, 561.

Cox, R. G. 1964 The motion of solid particles in viscous fluids. Ph.D. Thesis, Cambridge University.

Proudman, I. & Pearson, J. R. A. 1957 *J. Fluid Mech.* **2**, 237.

Rubinow, S. I. & Keller, J. B. 1961 *J. Fluid Mech.* **11**, 447.